ASYMPTOTIC LOWER BOUNDS FOR A CLASS OF SCHRÖDINGER EQUATIONS

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Abstract

We shall study the following initial value problem:

$$\mathbf{i}\partial_t u - \Delta u + V(x)u = 0,\ (t,x) \in \mathbf{R}\times\mathbf{R}^n, \tag{0.1}$$
$$u(0) = f,$$

where $V(x)$ is a real short–range potential, whose radial derivative satisfies some supplementary assumptions. More precisely we shall present a family of identities satisfied by the solutions to (0.1) that generalizes the ones proved in [12] and [21] in the free case. As a by–product of these identities we deduce some uniqueness results for solutions to (0.1), and a lower bound for the so called local smoothing which becomes an identity in a precise asymptotic sense.

1. Introduction

We shall study the following initial value problem:

$$\mathbf{i}\partial_t u - \Delta u + V(x)u = 0,\ (t,x) \in \mathbf{R}\times\mathbf{R}^n, \tag{1.1}$$
$$u(0) = f$$

under suitable assumptions on $V(x)$.

Let us recall that if $V(x) \in L^\infty(\mathbf{R}^n)$ then the operator

$$L^2(\mathbf{R}^n) \supset H^2(\mathbf{R}^n) \ni u \to -\Delta u + V(x)u \in L^2(\mathbf{R}^n)$$

is self–adjoint (see [14] for the proof of this fact). In particular one can apply the classical Stone theorem in order to deduce the existence of a unique solution $u(t,x) \in \mathcal{C}_t(L^2(\mathbf{R}^n))$ (here and in the sequel we shall denote by $\mathcal{C}_t(X)$ the space of continuous functions of one variable valued in the Banach space X) to the Cauchy problem (1.1), provided that $f \in L^2(\mathbf{R}^n)$.

Hereafter we shall denote by $e^{\mathbf{i}t\Delta_V} f$ the unique solution to (1.1) at time $t \in \mathbf{R}$. Let us recall that the following conservation law is satisfied:

$$\|e^{\mathbf{i}t\Delta_V} f\|_{L^2(\mathbf{R}^n)} \equiv \|f\|_{L^2(\mathbf{R}^n)}\ \forall t \in \mathbf{R}. \tag{1.2}$$

Notice that this identity implies that the operators $e^{\mathbf{i}t\Delta_V}$ define a family of isometries on $L^2(\mathbf{R}^n)$. Moreover, as a by–product of the Stone theorem one can deduce the following implication:

$$f \in H^2(\mathbf{R}^n) \Rightarrow e^{\mathbf{i}t\Delta} f \in \mathcal{C}^1_t(H^2(\mathbf{R}^n)) \tag{1.3}$$

(here we have denoted by $\mathcal{C}^1_t(X)$ the space of functions of one variable valued in the Banach space X with a continuous derivative). It is also well–known that the following conservation law holds:

$$\int_{\mathbf{R}^n} (|\nabla_x u(t,x)|^2 + V(x)|u(t,x)|^2)\ dx \tag{1.4}$$
$$= \int_{\mathbf{R}^n} (|\nabla_x f(x)|^2 + V(x)|f(x)|^2)\ dx\ \forall t \in \mathbf{R},$$

and in particular

$$\int_{\mathbf{R}^n} |\nabla_x u(t,x)|^2\ dx \leq C \int_{\mathbf{R}^n} (|\nabla_x f(x)|^2 + |f(x)|^2)\ dx\ \forall t \in \mathbf{R} \tag{1.5}$$

provided that $V(x) \geq 0$ and $V(x) \in L^\infty(\mathbf{R}^n)$.

In the sequel we shall assume that $V(x)$ satisfies the following decay assumption:

$$0 \leq V(x) \leq \frac{C}{(1+|x|)^{1+\epsilon}}\ \forall x \in \mathbf{R}^n \tag{1.6}$$

where $\epsilon, C > 0$.

We shall also assume either that $V(x)$ is decreasing in the radial variable, i.e.

$$\partial_{|x|} V \leq 0, \tag{1.7}$$

or that

$$\lim_{|x|\to\infty} |x|\partial_{|x|} V(x) = 0. \tag{1.8}$$

We shall specify in every theorem which kind of assumptions we assume on the derivative of V.

In order to state our results let us introduce the perturbed Sobolev spaces $\dot{H}^s_V(\mathbf{R}^n)$, whose norm is defined as follows:

$$\|f\|_{\dot{H}^s_V(\mathbf{R}^n)} \equiv \left\| \left(\sqrt{-\Delta + V}\right)^s u \right\|_{L^2(\mathbf{R}^n)}\ \forall s \geq 0. \tag{1.9}$$

Our first result contains a family of identities satisfied by solutions to (1.1). Let us underline that these identities represent a generalization of the identities proved in the free case, i.e. $V(x) \equiv 0$, in [12] and [21]. In fact in [22] a similar family of identities has been proved for the solutions to the conformally invariant nonlinear Schrödinger equation. Next we shall denote by $D^2\psi$ the Hessian matrix of the function ψ.

Theorem 1.1. *Let $u(t,x)$ be the solution to* (1.1) *where $n \geq 1$, $f \in C^\infty_0(\mathbf{R}^n)$. Assume moreover that $V(x)$ satisfies* (1.6) *and one of the conditions* (1.7) *or* (1.8)*. Let ψ be a radially symmetric function such that the following limit exists*

$$\lim_{|x|\to\infty} \partial_{|x|}\psi = \psi'(\infty) \in [0,\infty) \tag{1.10}$$

and moreover

$$\nabla\psi, D^2\psi, \Delta^2\psi \in L^\infty(\mathbf{R}^n)$$

Then the following identity holds:

$$\lim_{T\to\infty}\int_{-T}^{T}\int_{\mathbf{R}^n}\left[\nabla_x\bar{u}D^2\psi\nabla_x u-(\Delta^2\psi+4\partial_{|x|}V\partial_{|x|}\psi)\frac{|u|^2}{4}\right]dxdt \tag{1.11}$$
$$=\psi'(\infty)\|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^n)}.$$

As a by–product of the argument involved in the proof of theorem 1.1 we can construct a natural Banach space $\Sigma^{\frac{1}{2}}$ (whose definition will be given below) that is invariant along the flow associated to (1.1). Moreover we shall deduce one uniqueness result for solutions to (1.1) provided that $f\in\Sigma^{\frac{1}{2}}$. In order to define the space $\Sigma^{\frac{1}{2}}$ we first introduce the weighted Lebesgue space $L^2_{|x|}(\mathbf{R}^n)$ defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the following norm:

$$\|f\|^2_{L^2_{|x|}}\equiv\int_{\mathbf{R}^n}|x||f(x)|^2\ dx. \tag{1.12}$$

The Banach space $\Sigma^{\frac{1}{2}}$ is defined as follows:

$$\Sigma^{\frac{1}{2}}\equiv\dot{H}_V^{\frac{1}{2}}\cap L^2_{|x|} \tag{1.13}$$

and can be endowed with the norm

$$\|f\|^2_{\Sigma^{\frac{1}{2}}}\equiv\|f\|^2_{\dot{H}_V^{\frac{1}{2}}}+\|f\|^2_{L^2_{|x|}}.$$

We can state our second result.

Theorem 1.2. *Let $u(t,x)$ be the solution to (1.1) where $n\geq 2$, $f\in\Sigma^{\frac{1}{2}}$ and $V(x)$ satisfies the same assumptions as in theorem 1.1. Then we have the following a–priori estimate:*

$$\|u(t)\|^2_{\Sigma^{\frac{1}{2}}}\leq\|f\|^2_{L^2_{|x|}}+C(1+|t|)\|f\|^2_{\dot{H}_V^{\frac{1}{2}}}\ \forall t\in\mathbf{R}, \tag{1.14}$$

for a suitable $C>0$. In particular for every $t\in\mathbf{R}$ we have that $e^{\mathbf{i}t\Delta_V}f\in\Sigma^{\frac{1}{2}}$ provided that $f\in\Sigma^{\frac{1}{2}}$. Moreover

$$\lim_{t\to\pm\infty}\int_{\mathbf{R}^n}\frac{|x|}{|t|}|u(t,x)|^2\ dx=2\|f\|^2_{\dot{H}_V^{\frac{1}{2}}}. \tag{1.15}$$

In particular if

$$\lim_{t\to\pm\infty}\int_{\mathbf{R}^n}\frac{|x|}{|t|}|u(t,x)|^2\ dx=0$$

then $u\equiv 0$.

Remark 1.1. From a technical point of view we assume $n\geq 2$ in theorem 1.2, since the proof of lemma 4.1 (that in turn is needed in the proof of theorem 1.2) does not work in dimension $n=1$.

Remark 1.2. Along the proof of theorems 1.1 and 1.2, we shall make extensively use of the existence and completeness of the wave operator under the assumptions (1.6) and (1.8) on $V(x)$ (see section 2).

Remark 1.3. In order to prove theorems 1.1 and 1.2 we shall need some intemediate results, whose proof in some cases could be deduced by avoiding the use of the existence and completeness of the wave operator. For instance lemma 2.1 in section 2 follows from the general RAGE theory (see [15]). However we have proposed a proof that involves the existence and completeness of the wave operator in order to make the paper selfcontained as much as possible. In the appendix 7 we shall make some connections between the classical RAGE theorem and our results.

Next we shall deduce some direct consequences from the identity (1.11). In particular we shall show how it allows us to prove a lower bound to the classical local smoothing estimate. For a proof of the local smoothing estimate in the free case see [5], [18], [20] and also their extensions in [3], [7], [17]. In particular in [17] the issue of the best constants involved in the local smoothing estimate is considered.

First we shall present our results in dimension $n \geq 4$.

Theorem 1.3. *Let* $u(t,x)$ *be the solution to* (1.1)*, where* $n \geq 4$ *and* $V(x)$ *satisfies* (1.6) *and* (1.7)*. Then the following a–priori estimate is satisfied for every* $f \in \dot{H}^{\frac{1}{2}}_V(\mathbf{R}^n)$*:*

$$\|f\|^2_{\dot{H}^{\frac{1}{2}}_V(\mathbf{R}^n)} \leq \sup_{R>0} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\nabla_x u|^2 dx dt \leq C \|f\|^2_{\dot{H}^{\frac{1}{2}}_V(\mathbf{R}^n)}, \tag{1.16}$$

where $C > 0$ *is a suitable constant independent of* f*.*

In next result we give a better lower bound than the one in (1.16).

Theorem 1.4. *Let* $u(t,x)$ *be the solution to* (1.1) *where* $n \geq 4$*,* f *and* $V(x)$ *are as in theorem 1.3. Then we have:*

$$\lim_{R \to \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 dx dt = \|f\|^2_{\dot{H}^{\frac{1}{2}}_V(\mathbf{R}^n)}. \tag{1.17}$$

Therefore if

$$\lim_{R \to \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} |\partial_{|x|} u|^2 dx dt = 0$$

then $u \equiv 0$*.*

Remark 1.4. Notice that if you choose formally $\psi \equiv |x|$ in (1.11) and if you work in dimension $n \geq 4$, then the general identity (1.11) becomes

$$\int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \left(\frac{|\nabla_\tau u|^2}{|x|} + \frac{(n-1)(n-3)}{4} \frac{|u|^2}{|x|^3} - \partial_{|x|} V |u|^2 \right) dx dt = \|f\|^2_{\dot{H}^{\frac{1}{2}}_V(\mathbf{R}^n)} \tag{1.18}$$

(here $\nabla_\tau u$ denotes the angular part of the gradient of u). Let us underline that in the case $V(x) \equiv 0$ the previous identity has been proved in [12] with a different approach. Moreover (1.18) represents a precised version of the result in [9] and [10], where (1.18) is stated as an inequality and not as an identity. Notice also that the function $\psi \equiv |x|$ does not satisfies all the assumptions required in theorem 1.1. However in order to make precise the argument involved in the proof of (1.18), it is sufficient to choose in (1.11) the test function ψ to be equal to $\sqrt{\epsilon^2 + |x|^2}$ and to get the limit in the corresponding identity as $\epsilon \to 0$. An alternative way to prove properly (1.18) it is to combine the proof of (1.11) with the argument used in [10] (in fact in [10] the integration by parts technique that we use in the proof of (1.11) is completely justified also when $\psi \equiv |x|$).

Remark 1.5. Let us point out that theorems 1.3 and 1.4 are stated in dimension $n \geq 4$, while theorem 1.1 is stated in any dimension $n \geq 1$. The main reason is that in order to take advantage of the identity (1.11) we shall choose the test function $\psi(x)$ in a suitable way. As it will be clear in the sequel, we shall be able to make such a good choice only in dimension $n \geq 4$.

In dimension $n = 3$ we are able to prove the following result.

Theorem 1.5. *Let $u(t,x)$ be the solution to* (1.1) *where $n = 3$, $V(x)$ satisfies* (1.6) *and* (1.7)*. Then the following a priori estimate is satisfied for every $f \in \dot{H}_V^{\frac{1}{2}}(\mathbf{R}^3)$:*

$$(1.19)\quad c\|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^3)} \leq \sup_{R>0} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} \left(|\nabla_x u|^2 + \frac{1}{R^2}|u|^2 \right) dxdt \leq C\|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^3)},$$

where $c, C > 0$ are constants independent of f.

Theorem 1.6. *Let $u(t,x)$ be the solution to* (1.1) *where $n = 3$, f and $V(x)$ are as in theorem 1.5. Then we have:*

$$(1.20)\qquad \liminf_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} \left(|\partial_{|x|} u|^2 dxdt + \frac{1}{R^2}|u|^2 \right) dxdt \geq \frac{1}{2}\|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^3)}.$$

Therefore if

$$\liminf_{R\to\infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{|x|<R} \left(|\partial_{|x|} u|^2 + \frac{1}{R^2}|u|^2 \right) dxdt = 0$$

then $u \equiv 0$.

Remark 1.6. Starting with (1.11) it is possible to show the following version of the identity (1.18) in dimension $n = 3$:

$$(1.21)\quad \int_{-\infty}^{\infty} \int_{\mathbf{R}^3} \left(\frac{|\nabla_\tau u|^2}{|x|} - \partial_{|x|} V |u|^2 \right) dxdt + \frac{3}{2}\pi \int_{-\infty}^{\infty} |u(0,t)|^2 dt = \|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^3)}.$$

Exactly as for (1.18) the previous identity has been proved previously in [12] in the free case and it represents a precised version of a result proved in [9] and [10], where (1.21) is stated as an inequality.

Indeed the proof of (1.21) follows formally by choosing the function $\psi \equiv |x|$ in (1.11). However in dimension $n = 3$ the function $\psi \equiv |x|$ is very singular since its bilaplacian is a multiple of the Dirac delta and hence in order to justify all the computations we have to argue as in dimension $n \geq 4$ (see remark 1.4).

The paper is organized as follows. In section 2 we shall prove some asymptotic properties of solutions to (1.1). Sections 3 and 4 will be devoted to the proof of theorems 1.1 and 1.2. The proof of theorems 1.3 and 1.4 will be given in section 5, while theorems 1.5 and 1.6 will be proved in section 6. Finally in the appendix 7 we shall discuss some connections between our results and the classical RAGE theorem.

Next we shall fix some notations.

Notations. For every potential $V(x) \geq 0$ and for every real number $s \geq 0$ we shall denote by $\dot{H}_V^s$ the perturbed Sobolev space whose norm is defined in (1.9).

In particular when $V \equiv 0$ these spaces reduce to the standard Sobolev spaces $\dot{H}^s$ whose norm is defined as follows

$$\|f\|^2_{\dot{H}^s} \equiv \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi,$$

where

$$\hat{f}(\xi) \equiv \int_{\mathbf{R}^n} e^{-2\pi \mathbf{i} x \xi} f(x) dx.$$

In some cases we shall also write

$$\mathcal{F}(f) \equiv \hat{f}.$$

The spaces $L^2_{|x|}$ and $\Sigma^{\frac{1}{2}}$ are respectively the ones defined in (1.12) and (1.13).

For any $1 \le p, q \le \infty$

$$L^p_x \text{ and } L^p_t L^q_x$$

denote the Banach spaces

$$L^p(\mathbf{R}^n) \text{ and } L^p(\mathbf{R}; L^q(\mathbf{R}^n)).$$

We shall also write

$$L^p_t L^p_x \equiv L^p_{t,x}.$$

For every $V(x) \in L^\infty_x$ we shall denote by $e^{\mathbf{i}t\Delta_V}$ the group associated to (1.1) via the Stone theorem.

Given any couple of Banach spaces X and Y, we shall denote by $\mathcal{L}(X, Y)$ the space of linear and continuous functionals between X and Y.

Given a space–time dependent function $w(t, x)$ we shall denote by $w(t_0)$ the trace of w at fixed time $t \equiv t_0$, in case that it is well–defined.

We shall denote by $\int ... \, dx$, $\int ... \, dt$ and $\int\int ... \, dxdt$ the integral of suitable functions with respect to space, time, and space–time variables respectively.

When it is not better specified we shall denote by ∇v the gradient of any time–dependent function $v(t, x)$ with respect to the space variables. Moreover ∇_τ and $\partial_{|x|}$ shall denote respectively the angular gradient and the radial derivative.

If $\psi \in C^2(\mathbf{R}^n)$, then $D^2\psi$ will represent the hessian matrix of ψ.

Given a set $A \subset \mathbf{R}^n$ we denote by χ_A its characteristic function.

We shall use the function

$$\langle x \rangle \equiv \sqrt{1 + |x|^2}.$$

2. Wave operators and asymptotic behaviour of solutions

Let us recall that if $V(x)$ satisfies (1.6), then the wave operators $\mathcal{W}_\pm$ are well–defined and complete (see [1], [6], [15] and [16]). More precisely for every $f \in L^2_x$ there exist two functions $\mathcal{W}_\pm(f) \in L^2_x$ uniquely defined and such that

$$\lim_{t \to \pm\infty} \|u(t) - e^{\mathbf{i}t\Delta} \mathcal{W}_\pm(f)\|_{L^2_x} = 0, \tag{2.1}$$

where $u \in \mathcal{C}_t(L^2_x)$ denotes the unique solution to (1.1) with initial data f and $e^{\mathbf{i}t\Delta}$ represents the propagator at time t associated to the free Schrödinger equation, i.e. (1.1) with $V(x) \equiv 0$ (for a proof of (2.1) see [1]).

In the sequel we shall need the following asymptotic description of the free waves, whose proof can be found in [14].

Proposition 2.1. *Assume $f \in L^2_x$ and $n \geq 1$, then:*

$$\lim_{t\to\pm\infty} \left\| e^{\mathbf{i}t\Delta} f - e^{\mp \mathbf{i} n\pi/4} \frac{e^{\pm \mathbf{i}\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \hat{f}\left(\pm \frac{x}{4\pi t}\right) \right\|_{L^2_x} = 0. \tag{2.2}$$

Since now on we shall denote by $\mathcal{W}_\pm$ the wave operators defined above and by $\mathcal{F}$ the Fourier transform.

Next we shall state one of the basic results of this paper.

Proposition 2.2. *Let $u(t,x)$ be the solution to (1.1) where $n \geq 1$, $f \in C_0^\infty(\mathbf{R}^n)$ and $V(x)$ satisfies (1.6) and one of the two conditions (1.7) or (1.8). Assume that ψ is a radially symmetric such that the following limit exists:*

$$\lim_{|x|\to\infty} \partial_{|x|}\psi = \psi'(\infty) \in [0,\infty). \tag{2.3}$$

Then

$$\lim_{t\to\pm\infty} \mathcal{I}m\left(\int \bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx\right) = \mp 2\pi\psi'(\infty)\int |x||g_\pm(x)|^2\ dx \tag{2.4}$$

where $g_\pm = \mathcal{F}[\mathcal{W}_\pm f]$. Moreover the following identity holds:

$$\|f\|^2_{\dot{H}^{\frac{1}{2}}_V} = 2\pi\int |x||g_\pm(x)|^2\ dx. \tag{2.5}$$

We shall need the following lemma which is a consequence of the RAGE theorem. In order to be self–contained, we have decided to include a proof of it which is based on the existence and completeness of the wave operators $\mathcal{W}_\pm$ introduced at the beginning of the section.

Lemma 2.1. *Let $u(t,x)$ be the solution to (1.1) where $n \geq 1$, $f \in L^2_x$ and $V(x)$ satisfies (1.6), then*

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t\int W(x)|u|^2\ dxds = 0 \tag{2.6}$$

where $W \in L^\infty_x$ is such that

$$\lim_{|x|\to\infty} W(x) = 0.$$

Proof. We shall prove the result only as $t\to\infty$, since the case $t\to-\infty$ can be treated similarly.

Notice the following identity

$$\int_0^t\int W(x)|u|^2\ dxds = I(t) + II(t)\ \forall t > 0 \tag{2.7}$$

where

$$I(t) = \int_0^t\int W(x)\left[|u|^2 - \frac{1}{(4\pi s)^n}\left|\hat{g}\left(\frac{x}{4\pi s}\right)\right|^2\right]\ dxds$$

and

$$II(t) = \int_0^t\int W(x)\left|\hat{g}\left(\frac{x}{4\pi s}\right)\right|^2\ \frac{dxds}{(4\pi s)^n}.$$

where $g \equiv \mathcal{W}_+(f)$. Due to (2.2) and to the assumption $W \in L^\infty_x$ we can deduce

$$I(t) \leq \|W\|_{L^\infty_x}\int_0^t h(s)ds \text{ and } \lim_{s\to\infty} h(s) = 0 \tag{2.8}$$

where

$$h(s)=\int\left[|u(s)|^2-\frac{1}{(4\pi s)^n}\left|\hat{g}\left(\frac{x}{4\pi s}\right)\right|^2\right]\,dx.$$

On the other hand we have

$$II(t)=\int_0^t\int W(4\pi sx)|\hat{g}(x)|^2\,dsdx$$

that due to the dominated convergence theorem and to the decay assumption made on $W(x)$ implies

$$II(t)=\int_0^t H(s)ds \text{ and } \lim_{s\to 0}H(s)=0 \tag{2.9}$$

where

$$H(s)=\int W(4\pi sx)|\hat{g}(x)|^2\,dx.$$

By combining (2.7) with (2.8) and (2.9) it is easy to deduce (2.6).

□

Remark 2.1. Notice that in order to deduce (2.6) we have shown that

$$\lim_{t\to\pm\infty}\int W(x)|u(t)|^2\,dx=0 \tag{2.10}$$

which is stronger than (2.6). In fact (2.6) could be proved for a much larger class of potentials $V(x)$ by using the general RAGE theorem. In the appendix 7 we shall show how to deduce (2.10) by using as a starting point (2.6).

Lemma 2.2. *Let $u(t,x)$ be the solution to* (1.1) *where $n\geq 1$, $f\in C_0^\infty(\mathbf{R}^n)$ and $V(x)$ satisfies* (1.6) *and one of the two conditions* (1.7) *or* (1.8). *Then we have:*

$$\lim_{t\to\pm\infty}\left\|\frac{x}{t}u(t)-2\mathbf{i}\nabla u(t)\right\|_{L^2_x}=0. \tag{2.11}$$

Proof. We prove (2.11) only for $t\to\infty$, since the case $t\to-\infty$ is similar.

First case: $V(x)$ satisfies (1.7)

The following identity is well–known (see [4]):

$$\|xu(t)-2\mathbf{i}t\nabla u(t)\|^2_{L^2_x}+4t^2\int V(x)|u(t)|^2dx \tag{2.12}$$

$$=\int|x|^2|f(x)|^2\,dx+\int_0^t s\theta(s)\,ds\ \forall t\in\mathbf{R}$$

where

$$\theta(s)=8\int\left(V(x)+\frac{1}{2}|x|\partial_{|x|}V(x)\right)|u(s)|^2\,dx.$$

By combining the sign assumption done on $V(x)$ and $\partial_{|x|}V$ with (2.12) we get:

$$\left\|\frac{x}{t}u(t)-2\mathbf{i}\nabla u(t)\right\|^2_{L^2_x} \tag{2.13}$$

$$\leq\frac{\int|x|^2|f(x)|^2\,dx}{t^2}+\frac{8\int_0^t\left(\int V(x)|u(s)|^2\,dx\right)\,ds}{t}.$$

By combining this inequality with (2.6) we get (2.11).

Second case: $V(x)$ satisfies (1.8)

In this case we can use (2.12) as above and we can deduce

$$\left\|\frac{x}{t}u(t)-2\mathbf{i}\nabla u(t)\right\|_{L^2_x}^2 \leq \frac{\int |x|^2|f(x)|^2\ dx}{t^2}+\frac{\int_0^t\left(\int W(x)|u(s)|^2\ dx\right)\ ds}{t} \tag{2.14}$$

where

$$W(x)=8\left(V(x)+\frac{1}{2}|x|\partial_{|x|}V(x)\right).$$

Notice that due to (1.6) and (1.8) we have that $\lim_{|x|\to\infty}W(x)=0$, then we can use (2.6) in order to deduce (2.11).

$\square$

Remark 2.2. Let us underline that in [6] it is proved the existence of a sequence $\{t_n\}_{n\in\mathbf{N}}$ such that:

$$\lim_{n\to\infty}t_n=\infty \text{ and } \lim_{n\to\infty}\left\|\frac{x}{t_n}u(t_n)-2\mathbf{i}\nabla u(t_n)\right\|_{L^2_x}=0. \tag{2.15}$$

Notice that (2.15) is a weaker version of (2.11), however in [6] it is a basic tool in order to prove the completeness of the wave operators, provided that $V(x)$ satisfies the assumptions (1.6) and (1.8).

Remark 2.3. Notice that we obtain (2.11) using lemma 2.1 which we prove using the completeness of the wave operators. In the appendix 7 we shall give a proof of (2.11) that does not involve a–priori the completeness of the wave operator. Moreover we shall show that (2.11) is still satisfied for a class of potentials $V(x)$ more general than the ones that satisfy the decay assumptions (1.6) and (1.8).

Proof of proposition 2.2 As usual we treat only the case $t\to\infty$, the case $t\to-\infty$ can be treated similarly.

Along the proof we shall use the function $g(x)$ defined as follows

$$g\equiv\mathcal{F}[\mathcal{W}_+(f)].$$

Let us introduce the following identity:

$$\mathcal{I}m\int\bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx=I(t,R)+II(t,R)\ \forall t\in\mathbf{R},R>0, \tag{2.16}$$

where

$$I(t,R)=\mathcal{I}m\left(\int_{|x|>4\pi Rt}\bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx\right)$$

and

$$II(t,R)=\mathcal{I}m\left(\int_{|x|<4\pi Rt}\bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx\right).$$

Estimate for $I(t,R)$

Notice that the Cauchy–Schwartz inequality implies:

$$\left|\int_{|x|>4\pi Rt}\bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx\right|\leq C\|\nabla\psi\|_{L^\infty_x}\|f\|_{H^1_x}\left(\int_{|x|>4\pi Rt}|u(t)|^2dx\right)^{\frac{1}{2}}. \tag{2.17}$$

where we have used (1.5).

On the other hand due to (2.2) and due to the definition of $g(x)$ we get:

$$\lim_{t\to\infty}\int_{|x|>4\pi Rt}\left[|u(t)|^2-\frac{1}{(4\pi t)^n}\left|g\left(\frac{x}{4\pi t}\right)\right|^2\right]dx=0$$

and then

$$\lim_{t\to\infty}\int_{|x|>4\pi Rt}|u(t)|^2dx=\int_{|x|>R}|g(x)|^2dx. \tag{2.18}$$

Since $g\in L^2_x$ we can combine (2.17) with (2.18) in order to deduce that

$$\forall\epsilon>0\ \exists R(\epsilon)>0 \text{ s. t.} \limsup_{t\to\infty}|I(t,R)|<\epsilon\ \forall R>R(\epsilon). \tag{2.19}$$

Estimate for $II(t,R)$

Notice that (1.2) and (2.11) imply:

$$\lim_{t\to\infty}\left[\int_{|x|<4\pi Rt}\bar{u}(t)\nabla u(t)\cdot\nabla\psi\ dx+\frac{\mathbf{i}}{2t}\int_{|x|<4\pi Rt}|x|\partial_{|x|}\psi|u(t)|^2\ dx\right]=0. \tag{2.20}$$

Moreover we have the following identities:

$$\begin{aligned}&\int_{|x|<4\pi Rt}|x|\partial_{|x|}\psi|u(t)|^2\ \frac{dx}{t} \qquad (2.21)\\ &=\int_{|x|<4\pi Rt}|x|\partial_{|x|}\psi\left[|u(t)|^2-\frac{1}{(4\pi t)^n}\left|g\left(\frac{x}{4\pi t}\right)\right|^2\right]\ \frac{dx}{t}\\ &+\frac{1}{(4\pi)^n}\int_{|x|<4\pi Rt}|x|\left((\partial_{|x|}\psi-\psi'(\infty)\right)\left|g\left(\frac{x}{4\pi t}\right)\right|^2\frac{dx}{t^{n+1}}\\ &+\frac{1}{(4\pi)^n}\psi'(\infty)\int_{|x|<4\pi Rt}|x|\left|g\left(\frac{x}{4\pi t}\right)\right|^2\frac{dx}{t^{n+1}}.\end{aligned}$$

Notice that the following estimate is trivial:

$$\begin{aligned}&\left|\int_{|x|<4\pi Rt}|x|\partial_{|x|}\psi\left[|u(t)|^2-\frac{1}{(4\pi t)^n}\left|g\left(\frac{x}{4\pi t}\right)\right|^2\right]\ \frac{dx}{t}\right| \qquad (2.22)\\ &\leq 4\pi R\|\partial_{|x|}\psi\|_{L^\infty_x}\int\left||u(t)|^2-\frac{1}{(4\pi t)^n}\left|g\left(\frac{x}{4\pi t}\right)\right|^2\right|dx\to 0 \text{ as } t\to\infty,\end{aligned}$$

where at the last step we have combined (2.1) with (2.2).

Moreover the change of variable formula implies:

$$\begin{aligned}&\frac{1}{(4\pi)^n}\left|\int_{|x|<4\pi Rt}|x|\left((\partial_{|x|}\psi-\psi'(\infty)\right)\left|g\left(\frac{x}{4\pi t}\right)\right|^2\frac{dx}{t^{n+1}}\right|\\ &\leq 4\pi R\int\left|\partial_{|x|}\psi(4\pi tx)-\psi'(\infty)\right||g(x)|^2dx,\end{aligned}$$

that in conjunction with the dominated convergence theorem and with assumption (2.3) implies:

$$\lim_{t\to\infty}\frac{1}{(4\pi)^n}\int_{|x|<4\pi Rt}|x|\left((\partial_{|x|}\psi-\psi'(\infty)\right)\left|g\left(\frac{x}{4\pi t}\right)\right|^2\frac{dx}{t^{n+1}}=0. \tag{2.23}$$

Due again to the change of variable formula we get

$$\frac{\psi'(\infty)}{(4\pi)^n}\int_{|x|<4\pi Rt}|x|\left|g\left(\frac{x}{4\pi t}\right)\right|^2\frac{dx}{t^{n+1}}=4\pi\psi'(\infty)\int_{|x|<R}|x||g(x)|^2dx,$$

and in particular

$$\lim_{t\to\infty}\frac{\psi'(\infty)}{(4\pi)^n}\int_{|x|<4\pi Rt}|x|\left|g\left(\frac{x}{4\pi t}\right)\right|^2\frac{dx}{t^{n+1}}=4\pi\psi'(\infty)\int_{|x|<R}|x||g(x)|^2\ dx. \tag{2.24}$$

By combining (2.22),(2.23), (2.24) with (2.20) and (2.21) we deduce

$$\lim_{t\to\infty}II(t,R)=-2\pi\psi'(\infty)\int_{|x|<R}|x||g(x)|^2dx. \tag{2.25}$$

By combining (2.16) with (2.19) and (2.25) we get (2.4) at least in the case $t\to\infty$. The other case is similar.

Proof of (2.5)

Recall that $\mathcal{W}_+ : L^2_x \to L^2_x$ is an isometry and moreover

$$\mathcal{W}_+\circ f(-\Delta)=f(-\Delta+V)\circ\mathcal{W}_+.$$

By combining these facts with the definition of g, i.e. $g\equiv\mathcal{F}[\mathcal{W}_+f]$, we get:

$$\int|x||g(x)|^2\ dx=\|\mathcal{W}_+f\|^2_{\dot H^{\frac12}_x}=\frac{1}{2\pi}\|(-\Delta)^{\frac14}\circ\mathcal{W}_+f\|^2_{L^2_x}$$
$$=\frac{1}{2\pi}\|\mathcal{W}_+\circ(-\Delta_V)^{\frac14}f\|^2_{L^2_x}=\frac{1}{2\pi}\|(-\Delta_V)^{\frac14}f\|^2_{L^2_x}=\frac{1}{2\pi}\|f\|^2_{\dot H^{\frac12}_V}.$$

□

3. Proof of theorem 1.1 and some consequences

Proof of theorem 1.1 Following [2] we multiply (1.1) by the quantity

$$\nabla\bar u\cdot\nabla\psi+\frac12\bar u\ \Delta\psi, \tag{3.1}$$

and we integrate on the strip $(-T,T)\times\mathbf{R}^n$. In this way we get the following family of identities:

$$\int_{-T}^{T}\int\left(\nabla\bar uD^2\psi\nabla u-\Delta^2\psi\frac{|u|^2}{4}-\partial_{|x|}V\partial_{|x|}\psi|u|^2\right)dxdt \tag{3.2}$$
$$=-\frac12\mathcal{I}m\sum_{\pm}\int\bar u(\pm T)\nabla u(\pm T)\cdot\nabla\psi\ dx,$$

(for more details on this computation see [2] and [21]).

Indeed all the integration by parts involved in the proof of (3.2) can be completely justified by a density argument due to (1.3).

Notice that the identity (1.11) follows by combining (2.4), (2.5) and (3.2).

□

Next we shall exploit (1.11) in order to deduce some a–priori estimates satisfied by the solutions to (1.1).

Lemma 3.1. *Assume that* $u(t,x)$ *solves* (1.1) *where* $n\ge4$, $f\in\dot H^{\frac12}_V$, $V(x)$ *satisfies* (1.6) *and* (1.7), *then*

$$\int\int\frac{1}{\langle x\rangle^3}|u|^2\ dxdt<\infty \tag{3.3}$$

and

$$\int\int|\partial_{|x|}V||u|^2\ dxdt<\infty. \tag{3.4}$$

Proof. Choose in (1.11) the function $\psi(x) \equiv |x|$ that is clearly a radially symmetric and convex function. Moreover we have $\partial_{|x|}\psi \equiv 1$ and

$$\Delta^2(|x|) = -\frac{(n-1)(n-3)}{|x|^3} \ \forall x \in \mathbf{R}^n \text{ where } n \geq 4. \tag{3.5}$$

Hence by choosing $\psi(x) \equiv |x|$ in (1.11), it is easy to deduce that

$$\frac{(n-1)(n-3)}{4} \int\int \frac{|u|^2}{|x|^3}\, dxdt - \int\int \partial_{|x|}V|u|^2\, dxdt \leq \|f\|^2_{\dot{H}_V^{\frac{1}{2}}} \tag{3.6}$$

where $C > 0$ is a suitable constant, and hence we get easily (3.3) and (3.4).

□

Next we state a version of (3.4) in dimension $n = 3$.

Lemma 3.2. *Assume that $u(t,x)$ is solution to* (1.1) *with $n = 3$, $f \in \dot{H}_V^{\frac{1}{2}}$, $V(x)$ satisfies* (1.6) *and* (1.7), *then:*

$$\int\int |\partial_{|x|}V||u|^2\, dxdt < \infty. \tag{3.7}$$

Proof. The proof of (3.7) is identical to the proof of (3.4). Notice that by choosing in (1.11) the test function $\psi(x) \equiv |x|$ and arguing as in the proof of lemma 3.1 we get

$$\frac{3}{2}\pi \int |u(0,t)|^2 dt - \partial_{|x|}V|u|^2\, dxdt \leq \|f\|^2_{\dot{H}_V^{\frac{1}{2}}(\mathbf{R}^3)}$$

where we have used the property

$$-\Delta^2(|x|) = 6\pi\delta_0 \text{ on } \mathbf{R}^3. \tag{3.8}$$

□

We can now deduce the following

Proposition 3.1. *Assume that $u(t,x)$ is a solution to* (1.1) *with $n \geq 4$, $f \in \dot{H}_V^{\frac{1}{2}}$, $V(x)$ satisfies* (1.6) *and* (1.7), *then:*

$$\lim_{R\to\infty} \int\int |\Delta^2\phi_R||u|^2\, dxdt = 0, \tag{3.9}$$

where ϕ is a radially symmetric function such that

$$|\Delta^2\phi| \leq \frac{C}{\langle x\rangle^3} \ \forall x \in \mathbf{R}^n \tag{3.10}$$

and $\phi_R = R\phi\left(\frac{x}{R}\right)$.

Proof. Notice that (3.10) trivially implies

$$\int\int |\Delta^2\phi_R||u|^2\, dxdt \leq C \int\int \frac{|u|^2}{R^3 + |x|^3}\, dxdt \to 0 \text{ as } R \to \infty,$$

where we have combined the dominated convergence theorem with (3.3).

□

Proposition 3.2. *Assume that $u(t,x)$ is solution to* (1.1) *with $n \geq 3$, $f \in \dot{H}_V^{\frac{1}{2}}$, $V(x)$ satisfies* (1.6) *and* (1.7), *then:*

$$\lim_{R\to\infty} \int\int |\partial_{|x|}V||\partial_{|x|}\phi_R||u|^2\, dxdt = 0 \tag{3.11}$$

where ϕ is a radially symmetric function such that

$$\partial_{|x|}\phi(0) = 0, |\partial_{|x|}\phi| \leq C \ \forall x \in \mathbf{R}^n$$

and $\phi_R = R\phi\left(\frac{x}{R}\right)$.

Proof. It follows from the following identity:

$$\int\int |\partial_{|x|}V||\partial_{|x|}\phi_R||u|^2 \ dxdt \tag{3.12}$$
$$= \int\int |\partial_{|x|}V| \left|\partial_{|x|}\phi\left(\frac{x}{R}\right)\right| |u|^2 \ dxdt \to 0 \text{ as } R \to \infty,$$

where at the last step we have combined the dominated convergence theorem with (3.4) (or with (3.7) in the specific case $n = 3$) and with the assumption $\partial_{|x|}\phi(0) = 0$. □

4. Proof of theorem 1.2

We shall need the following lemma, whose proof in dimension $n \geq 3$ follows an argument in [2].

Lemma 4.1. *Assume that $n \geq 2$ and $h \in L^2_x \cap \dot{H}^1_x$, then we have the following inequality:*

$$\int \bar{h}(x) \ \nabla h(x) \cdot \frac{x}{|x|} \ dx \leq C\|h\|^2_{\dot{H}^{\frac{1}{2}}_x}, \tag{4.1}$$

where $C > 0$ is a constant that depends on n.

Proof. We introduce the quadratic form

$$a(f, g) = \int \bar{f}(x) \ \nabla g(x) \cdot \frac{x}{|x|} \ dx.$$

Notice that the Cauchy–Schwartz inequality implies:

$$|a(f, g)| \leq \|f\|_{L^2_x}\|g\|_{\dot{H}^1_x}. \tag{4.2}$$

Next we split the proof in two cases.

First case: $n \geq 3$

By assuming f, g regular enough we can use integration by parts in order to deduce:

$$a(f, g) = -\int g(x)\nabla \bar{f}(x) \cdot \frac{x}{|x|} \ dx - (n-1)\int \frac{1}{|x|}\bar{f}(x)g(x) \ dx.$$

By combining the Hardy inequality and the Cauchy–Schwartz inequality with the previous identity we get:

$$|a(f, g)| \leq C\|f\|_{\dot{H}^1_x}\|g\|_{L^2_x} \tag{4.3}$$

Notice that (4.1) will follow by interpolation from (4.2) and (4.3) and by choosing $f = g = h$.

Second case: $n = 2$

In this case we are not allowed to use the Hardy inequality in order to deduce (4.3). Hence we shall look for a substitute of this inequality in dimension $n = 2$.

Due to the Parseval identity we get

$$\int \bar{f}(x)\partial_j g(x)\frac{x_j}{|x|} \ dx = \int |D|^{\frac{1}{2}}\left(\bar{f}(x)\frac{x_j}{|x|}\right)|D|^{-\frac{1}{2}}\left(\partial_j g(x)\right) \ dx,$$

where $|D| \equiv \sqrt{-\Delta}$ and $\partial_j \equiv \frac{\partial}{\partial x_j}$, and then due to the Cauchy–Schwartz inequality we deduce

$$\int \bar{f}(x)\partial_j g(x)\frac{x_j}{|x|}\,dx \leq \left\||D|^{\frac{1}{2}}\left(\bar{f}(x)\frac{x_j}{|x|}\right)\right\|_{L^2_x}\||D|^{\frac{1}{2}}g\|_{L^2_x} \tag{4.4}$$

$$= \left\||D|^{\frac{1}{2}}\left(\bar{f}(x)\frac{x_j}{|x|}\right)\right\|_{L^2_x}\|g\|_{\dot{H}^{\frac{1}{2}}_x}.$$

On the other hand we have the following chain of inequalities:

$$\left\||D|^{\frac{1}{2}}\left(\bar{f}(x)\frac{x_j}{|x|}\right)\right\|_{L^2_x} \leq \left\||D|^{\frac{1}{2}}\left(\bar{f}(x)\frac{x_j}{|x|}\right) - \bar{f}|D|^{\frac{1}{2}}\left(\frac{x_j}{|x|}\right)\right\|_{L^2_x} \tag{4.5}$$

$$+\left\|\bar{f}|D|^{\frac{1}{2}}\left(\frac{x_j}{|x|}\right)\right\|_{L^2_x} \leq \|f\|_{\dot{H}^{\frac{1}{2}}_x} + \left\|\bar{f}|D|^{\frac{1}{2}}\left(\frac{x_j}{|x|}\right)\right\|_{L^2_x}$$

where we have used the inequality

$$\||D|^s(fg) - f(|D|^s g)\|_{L^2_x} \leq \|g\|_{L^\infty_x}\||D|^s f\|_{L^2_x}$$

(for a proof see [8]).

On the other hand the following inequality can be proved:

$$\left||D|^{\frac{1}{2}}\left(\frac{x_j}{|x|}\right)\right| \leq \frac{C}{|x|^{\frac{1}{2}}},$$

(for a proof see for example [13]) and due to the Sobolev embedding it implies

$$\left\|\bar{f}|D|^{\frac{1}{2}}\left(\frac{x_j}{|x|}\right)\right\|_{L^2_x} \leq C\left\|\frac{1}{|x|^{\frac{1}{2}}}\bar{f}\right\|_{L^2_x} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}_x}.$$

By combining this inequality with (4.5) we get:

$$\left\||D|^{\frac{1}{2}}\left(\bar{f}(x)\frac{x_j}{|x|}\right)\right\|_{L^2_x} \leq C\|f\|_{\dot{H}^{\frac{1}{2}}_x},$$

that in turn with (4.4) gives

$$|a(f,g)| \leq \|f\|_{\dot{H}^{\frac{1}{2}}_x}\|g\|_{\dot{H}^{\frac{1}{2}}_x}.$$

The proof is complete.

□

Lemma 4.2. *Let $u(t,x)$ be the solution to (1.1) where $n \geq 1$ and $V(x)$ satisfies the same assumptions as in theorem 1.1, then the following a –priori estimates are satisfied:*

$$\|u(t)\|^2_{\dot{H}^{\frac{1}{2}}_x} \leq \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}_V} \leq \|f\|^2_{\dot{H}^{\frac{1}{2}}_V}\ \forall t \in \mathbf{R}. \tag{4.6}$$

Moreover we have

$$\|u(t)\|^2_{L^2_x} + \|u(t)\|^2_{\dot{H}^1_x} \leq C(\|f\|^2_{L^2_x} + \|f\|^2_{\dot{H}^1_x})\ \forall t \in \mathbf{R}. \tag{4.7}$$

Proof. Due to (1.2) and (1.4) we get:

$$\|u(t)\|_{L^2_x} = \|f\|_{L^2_x} \text{ and } \|u(t)\|_{\dot{H}^1_V} = \|f\|_{\dot{H}^1_V}.$$

Hence the r.h.s. in (4.6) follows by interpolation (see [19]).

Next notice that by hypothesis $V(x) \geq 0$ and then

$$\|h\|^2_{\dot{H}^1_x} \leq \int (|\nabla h(x)|^2 + V(x)|h(x)|^2) \ dx = \|h\|^2_{\dot{H}^1_V}.$$

Hence the l.h.s. in (4.6) will follow again from an interpolation argument.

The proof of (4.7) follows from (1.2) and (1.5).

□

Proof of theorem 1.2. Due to the r.h.s. in (4.6) it is easy to verify that (1.14) will follow from the following inequality:

$$\int |x||u(t)|^2 \ dx \leq \int |x||f(x)|^2 \ dx + C|t|\|f\|^2_{\dot{H}^{\frac{1}{2}}_V}. \tag{4.8}$$

In the sequel we shall prove (4.8) only for $t > 0$ (the proof is similar in the case $t < 0$).

Proof of (4.8) *for* $t > 0$

Since $u(t, x)$ solves (1.1) we have the following identity:

$$\frac{d}{dt}\int |x||u(t)|^2 \ dx = \int |x|(\partial_t u(t)\bar{u}(t) + u(t)\partial_t \bar{u}(t)) \ dx \tag{4.9}$$

$$= 2\mathcal{I}m \int |x|\Delta u(t)\bar{u}(t) \ dx = -2\mathcal{I}m \int \bar{u}(t)\nabla u(t) \cdot \frac{x}{|x|} \ dx$$

where we have used integration by parts.

In order to simplify the notation we introduce the function

$$G(t) \equiv -2\mathcal{I}m \int \bar{u}(t)\nabla u(t) \cdot \frac{x}{|x|} \ dx \tag{4.10}$$

that due to (4.1) and (4.6) satisfies:

$$|G(t)| \leq C\|f\|^2_{\dot{H}^{\frac{1}{2}}_V} \ \forall t \in \mathbf{R} \tag{4.11}$$

(notice that in fact in order to justify this computation we have to work by density and first we assume $f \in L^2_x \cap \dot{H}^1_x$. In this way due to (4.7) we have that $u(t) \in L^2_x \cap \dot{H}^1_x$ and hence we are in position to apply (4.1) with $h = u(t)$).

By combining this inequality with (4.9) we get

$$\int |x||u(t)|^2 \ dx \leq \int |x||u(0)|^2 \ dx + Ct\|f\|^2_{\dot{H}^{\frac{1}{2}}_V}$$

$$= \int |x||f(x)|^2 \ dx + Ct\|f\|^2_{\dot{H}^{\frac{1}{2}}_V} \ \forall t > 0.$$

Proof of (1.15)

We shall prove (1.15) only in the case $t \to \infty$ (the case $t \to -\infty$ can be treated in a similar way). Notice that (4.9) implies:

$$\int \frac{|x|}{t}|u(t)|^2 \ dx = \int \frac{|x|}{t}|f(x)|^2 \ dx + \frac{\int_0^t G(s) \ ds}{t} \ \forall t > 0. \tag{4.12}$$

On the other hand (2.4) and (2.5) (where we choose $\psi \equiv |x|$) imply

$$\lim_{t \to \pm\infty} G(t) = \pm 2\|f\|^2_{\dot{H}^{\frac{1}{2}}_V}. \tag{4.13}$$

By combining (4.11), (4.12) and (4.13) we finally get

$$\lim_{t\to\infty}\int \frac{|x|}{t}|u(t)|^2\ dx = 2\|f\|^2_{\dot{H}_V^{\frac{1}{2}}}. \tag{4.14}$$

□

5. Proof of theorems 1.3 and 1.4

We split the proof in two steps.

Proof of r.h.s. in (1.16)

It is sufficient to consider the identity (1.11) where the generic function ψ is replaced with the family of rescaled functions $R\phi\left(\frac{x}{R}\right)$ and $\phi(x) \equiv \langle x\rangle$. In fact notice that the function $\langle x\rangle$ is convex, increasing and moreover

$$-\Delta^2(\langle x\rangle) \geq 0\ \forall x \in \mathbf{R}^n, n \geq 3$$

as a direct computation shows.

Let us point–out that the l.h.s. in (1.16) follows from theorem 1.4.

Proof of theorem 1.4

Next we shall make use of the following identity

$$\nabla \bar{u} D^2\psi \nabla u = \partial^2_{|x|}\psi|\partial_{|x|}u|^2 + \frac{\partial_{|x|}\psi}{|x|}|\nabla_\tau u|^2, \tag{5.1}$$

where ψ is a radially symmetric function and u is a generic function.
First of all let us notice that if we choose in the identity (1.11) the function $\psi \equiv |x|$ then we get:

$$\int\int_{|x|>1} \frac{|\nabla_\tau u|^2}{|x|}\ dxdt < \infty,$$

where we have used (3.5) and (5.1).

In particular we get

$$\lim_{R\to\infty}\int\int_{|x|>R} \frac{|\nabla_\tau u|^2}{|x|}\ dx = 0. \tag{5.2}$$

For any $k \in \mathbf{N}$ we fix a function $h_k(r) \in C_0^\infty(\mathbf{R};[0,1])$ such that:

$$h_k(r) = 1\ \forall r \in \mathbf{R} \text{ s.t. } |r| < 1, h_k(r) = 0\ \forall r \in \mathbf{R} \text{ s.t. } |r| > \frac{k+1}{k}, \tag{5.3}$$

$$h_k(r) = h_k(-r)\ \forall r \in \mathbf{R}.$$

Let us introduce the functions $\psi_k(r), H_k(r) \in C^\infty(\mathbf{R})$:

$$\psi_k(r) = \int_0^r (r-s)h_k(s)ds \ \text{ and }\ H_k(r) = \int_0^r h_k(s)ds. \tag{5.4}$$

Notice that

$$\psi_k''(r) = h_k(r), \psi_k'(r) = H_k(r)\forall r \in \mathbf{R} \text{ and } \lim_{r\to\infty}\partial_r\psi_k(r) = \int_0^\infty h_k(s)ds. \tag{5.5}$$

Moreover an elementary computation shows that:

$$\Delta^2\psi_k(x) = \frac{C}{|x|^3}\ \forall x \in \mathbf{R}^n \text{ s.t. } |x| \geq 2 \text{ and } n \geq 4, \tag{5.6}$$

where Δ^2 is the bilaplacian operator.

Thus the functions $\phi = \psi_k$ satisfy the assumptions of propositions 3.1 and 3.2.

In the sequel we shall need the rescaled functions

$$\psi_{k,R}(x) = R\psi_k\left(\frac{x}{R}\right) \forall x \in \mathbf{R}^n, k \in \mathbf{N} \text{ and } R > 0, \tag{5.7}$$

where ψ_k is defined in (5.4). Notice that by combining the general identity (5.1) with (1.11), where we choose $\psi = \psi_{k,R}$ defined in (5.7), and recalling (5.5) we get:

$$\int\int\left[\partial^2_{|x|}\psi_{k,R}|\partial_{|x|}u|^2 + \frac{\partial_{|x|}\psi_{k,R}}{|x|}|\nabla_\tau u|^2 \right. \tag{5.8}$$

$$\left. -\left(\frac{1}{4}\Delta^2\psi_{k,R} + \partial_{|x|}V\partial_{|x|}\psi_{k,R}\right)|u|^2\right]dxdt = \left(\int_0^\infty h_k(s)ds\right)\|f\|^2_{\dot{H}^{\frac{1}{2}}_V} \forall k \in \mathbf{N}, R > 0.$$

By using (3.9), (3.11) and (5.2) we get:

$$\lim_{R\to\infty}\int\int\left[\partial_{|x|}\psi_{k,R}\frac{|\nabla_\tau u|^2}{|x|} - \left(\frac{1}{4}\Delta^2\psi_{k,R} + \partial_{|x|}V\partial_{|x|}\psi_{k,R}\right)|u|^2\right]dxdt = 0$$

for every $k \in \mathbf{N}$.

We can combine this fact with (5.8) in order to deduce:

$$\lim_{R\to\infty}\int\int\partial^2_{|x|}\psi_{k,R}|\partial_{|x|}u|^2dxdt = \left(\int_0^\infty h_k(s)ds\right)\|f\|^2_{\dot{H}^{\frac{1}{2}}_V} \forall k \in \mathbf{N}. \tag{5.9}$$

On the other hand, due to the properties of h_k (see (5.3)), we get

$$\frac{1}{R}\int\int_{B_R}|\partial_{|x|}u|^2dxdt \leq \int\int\partial^2_{|x|}\psi_{k,R}|\partial_{|x|}u|^2dtdx$$

$$= \frac{1}{R}\int\int h_k\left(\frac{x}{R}\right)|\partial_{|x|}u|^2dtdx \leq \frac{1}{R}\int\int_{|x|<\frac{k+1}{k}R}|\partial_{|x|}u|^2dxdt$$

that due to (5.9) implies:

$$\limsup_{R\to\infty}\frac{1}{R}\int\int_{|x|<R}|\partial_{|x|}u|^2dxdt \leq \left(\int_0^\infty h_k(s)ds\right)\|f\|^2_{\dot{H}^{\frac{1}{2}}_V} \tag{5.10}$$

$$\leq \frac{k+1}{k}\liminf_{R\to\infty}\frac{1}{R}\int\int_{|x|<R}|\partial_{|x|}u|^2dxdt \ \forall k \in \mathbf{N}.$$

Since $k \in \mathbf{N}$ is arbitrary and since the following identity is trivially satisfied:

$$\lim_{k\to\infty}\int_0^\infty h_k(s)ds = 1,$$

we can deduce easily (1.17) by using (5.10).

The proof is complete.

□

6. Proof of theorems 1.5 and 1.6

The proofs are similar in principle to the ones of theorems 1.3 and 1.4, except that in dimension $n = 3$ we cannot use (3.3) and hence lemma 3.1, that have been proved only in dimension $n \geq 4$. However for completeness we shall give in the sequel the details of the proof in dimension $n = 3$.

We split the proof in two steps.

Proof of r.h.s. in (1.19)

It is sufficient to consider the identity (1.11) where the generic function ψ is replaced with the family of rescaled functions $R\phi\left(\frac{x}{R}\right)$ and $\phi(x) \equiv \langle x \rangle$. In fact notice that the function $\langle x \rangle$ is convex, increasing and moreover

$$-\Delta^2(\langle x \rangle) \geq 0 \ \forall x \in \mathbf{R}^3$$

as a direct computation shows.

Let us point–out that the l.h.s. in (1.19) follows from theorem 1.6.

Proof of theorem 1.6

Following the proof of (5.2) we get:

$$\lim_{R\to\infty} \int \int_{|x|>R} \frac{|\nabla_\tau u|^2}{|x|} \, dxdt = 0, \tag{6.1}$$

(in this case of course we have to use (3.8) instead of (3.5)).

We fix a function $h(r) \in C_0^\infty(\mathbf{R}; [0,1])$ such that:

$$h(r) = 1 \ \forall r \in \mathbf{R} \text{ s.t. } |r| < \frac{1}{2}, h(r) = 0 \ \forall r \in \mathbf{R} \text{ s.t. } |r| > 1,$$
$$h(r) = h(-r) \ \forall r \in \mathbf{R}.$$

Let us introduce the functions $\psi(r), H(r) \in C^\infty(\mathbf{R})$:

$$\psi(r) = \int_0^r (r-s)h(s)ds \ \text{ and } \ H(r) = \int_0^r h(s)ds. \tag{6.2}$$

Notice that

$$\psi''(r) = h(r), \psi'(r) = H(r) \forall r \in \mathbf{R} \text{ and } \lim_{r\to\infty} \partial_r \psi(r) = \int_0^\infty h(s)ds. \tag{6.3}$$

Moreover an elementary computation shows that:

$$\Delta^2 \psi(x) = 0 \ \forall x \in \mathbf{R}^3 \text{ s.t. } |x| \geq 1, \tag{6.4}$$

where Δ^2 is the bilaplacian operator (recall that we are working in dimension $n = 3$) and ψ is defined in (6.2)

Notice also that the function ψ given above satisfies the assumptions of proposition 3.2. In the sequel we shall need the rescaled functions

$$\psi_R(x) = R\psi\left(\frac{x}{R}\right) \forall x \in \mathbf{R}^3 \text{ and } R > 0. \tag{6.5}$$

By combining the identity (5.1) with (1.11), where we choose $\psi = \psi_R$, and recalling (6.3) we get:

$$\int \int \Big[\partial^2_{|x|} \psi_R |\partial_{|x|} u|^2 + \frac{\partial_{|x|} \psi_R}{|x|} |\nabla_\tau u|^2 \tag{6.6}$$

$$-\left(\frac{1}{4}\Delta^2\psi_R+\partial_{|x|}V\partial_{|x|}\psi_R\right)|u|^2\Big]\,dxdt=\left(\int_0^\infty h(s)ds\right)\|f\|^2_{\dot H_V^{\frac12}}\,\forall R>0.$$

By using (3.11) and (6.1) we get:

$$\lim_{R\to\infty}\int\int\left(\partial_{|x|}\psi_R\frac{|\nabla_\tau u|^2}{|x|}-\partial_{|x|}V\partial_{|x|}\psi_R|u|^2\right)dxdt=0.$$

We can combine this fact with (6.6) in order to deduce:

$$\lim_{R\to\infty}\int\int\left(\partial^2_{|x|}\psi_R|\partial_{|x|}u|^2-\frac{1}{4}\Delta^2\psi_R|u|^2\right)dxdt=\left(\int_0^\infty h(s)ds\right)\|f\|^2_{\dot H_V^{\frac12}}. \tag{6.7}$$

On the other hand, due to the definition of the functions $h(|x|)$ and $\psi_R(|x|)$ (see (6.2) and (6.5)), we get

$$\int\int\left(\partial^2_{|x|}\psi_R|\partial_{|x|}u|^2-\frac{1}{4}\Delta^2\psi_R|u|^2\right)dxdt$$
$$\leq\int\int\left(\frac{1}{R}h\left(\frac{x}{R}\right)|\partial_{|x|}u|^2-\frac{1}{4R^3}(\Delta^2\psi)^-\left(\frac{x}{R}\right)|u|^2\right)dxdt$$

(here $(\Delta^2\psi)^-$ represents the negative part of $\Delta^2\psi$), that in turn due to (6.4) and again to the property of the support of $h(|x|)$ implies:

$$\int\int\left(\partial^2_{|x|}\psi_R|\partial_{|x|}u|^2-\frac{1}{4}\Delta^2\psi_R|u|^2\right)dxdt$$
$$\leq\frac{1}{R}\int\int_{|x|<R}\left(|\partial_{|x|}u|^2+\frac{C}{R^2}|u|^2\right)dxdt.$$

Finally we can combine this estimate with (6.7) in order to deduce (1.20).

The proof is complete.

□

7. Appendix

In this section we shall work with potentials $V(x)\in L^\infty_x$ such that

$$\lim_{|x|\to\infty}V(x)=0\text{ and }\lim_{|x|\to\infty}|x||\partial_{|x|}V(x)|=0. \tag{7.1}$$

We point out that this appendix does not contain any essential novelty, since the content of proposition 7.1 in part follows from the results in [11] (see remark 7.1).

However, our aim in this appendix is to show in a very simple way how to deduce a stronger version of the usual RAGE theorem by using as a starting point the classical RAGE theorem himself (see proposition 7.1) and by avoiding the use of the general Mourre theorem in [11].

It is well–known that to every bounded potential $V(x)$, we can associate a corresponding splitting of the Hilbert space L^2_x as the direct sum of the projection onto the continuous spectrum and onto the pure point spectrum, that shall be denoted respectively as L^2_c and L^2_{pp}.

The space L^2_c can be splitted in turn as the direct sum of the projection onto the singular spectrum and onto the absolutely continuous spectrum, that shall be denoted respectively as L^2_s and L^2_{ac}.

Hereafter we shall make use of the following version of the RAGE theorem:

$$\text{if } u \in \mathcal{C}_t(L^2) \text{ solves (1.1) where } f \in L^2_c \tag{7.2}$$
$$\text{then } \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T}\int_{|x|<R} |u(t)|^2 \, dxdt = 0 \ \forall R > 0.$$

Actually the classical RAGE theorem is much more general than (7.2) (see [15]), however (7.2) will be enough for our purposes.

One of the aims of this appendix is to show how the RAGE theorem implies (2.11), under the decay assumptions on $V(x)$ given in (7.1). Let us recall that (2.11) is a stronger version of a result obtained in [6] (see remark 2.2).

Another point in this appendix is to show how it is possible to prove a decay of the solution u pointwisely in time, by using as a starting point the decay given in (7.2).

Next we state the main result of the section.

Proposition 7.1. *Let $V(x)$ be a function that satisfies* (7.1). *Assume that the point spectrum of $-\Delta + V$ is the empty set. Then we have:*

$$\lim_{t\to\pm\infty} \left\| \frac{x}{t} u(t) - 2\mathbf{i}\nabla u(t) \right\|_{L^2_x} = 0 \tag{7.3}$$

where

$$u(t) \equiv e^{\mathbf{i}t\Delta_V} f \ \textit{with} \ \int |x|^2 |f|^2 \, dx$$

and

$$\lim_{t\to\pm\infty} \int W(x)|u(t)|^2 \, dx = 0 \tag{7.4}$$

where

$$u(t) \equiv e^{\mathbf{i}t\Delta_V} f \ \textit{with} \ f \in L^2_x$$

and $W(x) \in L^\infty_x$ satisfies

$$\lim_{|x|\to\infty} W(x) = 0.$$

In particular we have:

$$e^{\mathbf{i}t\Delta_V} f \rightharpoonup 0 \ \textit{as} \ t \to \pm\infty \ \forall f \in L^2_x. \tag{7.5}$$

Remark 7.1. Looking at the proof of the RAGE theorem, which is based on Wiener's result about the decay of the Fourier transform of a measure, it is no difficult to show that (7.4) is trivially satisfied provided that the projection on the absolutely continuous spectrum L^2_{ac} coincides with L^2, or equivalently when the projection on the singular spectrum L^2_s is trivial. Actually this fact has been proved in [11], provided that the potential $V(x)$ satisfies (7.1). However we have decided to present our own proof of proposition 7.1 due to its simplicity.

Proof of proposition 7.1. For simplicity we shall treat only the limit as $t \to \infty$ (the case $t \to -\infty$ can be studied in a similar way).

Proof of (7.3)

We have that $L^2_x \equiv L^2_c$. Hence we can use the RAGE theorem (see (7.2)) in order to deduce that

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T}\int_{|x|<R} |u(t)|^2 \, dxdt = 0, \tag{7.6}$$

where $u(t) = e^{\mathbf{i}t\Delta_V} f$.

Notice that $u \equiv e^{\mathbf{i}t\Delta_V} f \in \mathcal{C}_t(L^2_x)$ can be also defined as the unique solution to

$$\mathbf{i}\partial_t u - \Delta u + V(x)u = 0,$$

$$u(0) = f,$$

hence we can use the estimate (2.14) given in lemma 2.2 in order to get:

$$\left\| \frac{x}{t} u(t) - 2\mathbf{i}\nabla u(t) \right\|^2_{L^2_x}$$

$$\leq \frac{\int |x|^2 |f(x)|^2 \, dx}{t^2} + \frac{8 \int_0^t \left(\int U(x)|u(s)|^2 \, dx \right) \, ds}{t}$$

where

$$U(x) \equiv 8 \left(V(x) + \frac{1}{2}|x|\partial_{|x|} V(x) \right).$$

Due to (7.6) and to the decay assumption (7.1), the previous estimate can be rewritten as follows:

$$\left\| \frac{x}{t} u(t) - 2\mathbf{i}\nabla u(t) \right\|^2_{L^2_x} \tag{7.7}$$

$$\leq \frac{\int |x|^2 |f(x)|^2 \, dx}{t^2} + \varphi(t)$$

where

$$\lim_{t \to \infty} \varphi(t) = 0.$$

Hence the proof of (7.3) follows easily.

Proof of (7.4)

Due to (1.2) and to a density argument, it is sufficient to prove (7.4) under the assumption $f \in C^\infty_0(\mathbf{R}^n)$.

Next notice that (7.7) can be written as

$$4 \left\| \nabla (e^{\mathbf{i}\frac{|x|^2}{4t}} u(t)) \right\|^2_{L^2_x} \leq \frac{\int |x|^2 |f(x)|^2 \, dx}{t^2} + \varphi(t) \tag{7.8}$$

where

$$\lim_{t \to \infty} \varphi(t) = 0.$$

Since we are assuming $f \in C^\infty_0(\mathbf{R}^n)$ we have that the R.H.S. in (7.8) goes to zero as $t \to \infty$.

Assume $n > 2$. By the Sobolev embedding

$$\dot{H}^1(\mathbf{R}^n) \subset L^{2^*}(\mathbf{R}^n) \text{ where } 2^* \equiv \frac{2n}{n-2},$$

we get

$$\lim_{t \to \infty} \|u(t)\|_{L^{2^*}} = 0. \tag{7.9}$$

This estimate in conjunction with the Hölder inequality implies:

$$\int_{|x|<R} |u(t)|^2 \, dx \leq CR^2 \|u(t)\|^2_{L^{2^*}} \ \forall R > 0,$$

that in turn can be combined with (7.9) in order to give

$$\lim_{t\to\infty}\int_{|x|<R}|u(t)|^2\,dx=0\ \forall R>0. \tag{7.10}$$

Notice that (7.4) follows by (7.10), (1.2) and the decay assumption of $W(x)$ at infinity.

Finally notice that (7.5) follows by combining (1.2) with (7.4).

For $n=1$ and $n=2$ a similar argument works using Gagliardo–Nirenberg inequalities instead of the Sobolev embedding.

□

The aim of the next proposition is to show that, despite to (7.4), in general there is a–priori no rate of decay for the L^2 localized norm of the solutions u to (1.1).

Proposition 7.2. *Assume that $V(x)\in L^\infty_x$ is any bounded potential (possibly $V(x)\equiv 0$). Let $R>0$ be a fixed positive number and $\gamma\in\mathcal{C}([0,\infty);\mathbf{R})$ be any function such that*

$$\lim_{t\to\infty}\gamma(t)=\infty.$$

Then there exists $g\in L^2_x$ (that depends on $R, V(x)$ and $\gamma(t)$)such that

$$\int_{|x|<R}|u(t_n)|^2\,dx>\frac{n}{\gamma(t_n)}$$

where $\{t_n\}_{n\in\mathbf{N}}$ is a suitable sequence $\lim_{n\to\infty}t_n=\infty$ and $u(t)\in\mathcal{C}_t(L^2_x)$ satisfies

$$\mathbf{i}\partial_t u-\Delta u+V(x)u=0, \tag{7.11}$$

$$u(0)=g.$$

Proof. We claim the following fact:

$$\|\mathcal{U}_{R,V}(t)\|_{\mathcal{L}(L^2_x,L^2_x)}\equiv 1\quad\forall t\in\mathbf{R}\text{ where} \tag{7.12}$$

$$\mathcal{U}_{R,V}(t):L^2_x\ni g\to\chi_{\{|x|<R\}}u(t)\in L^2_x$$

and $u(t)$ denotes the unique solution to the Cauchy problem (7.11).

Notice that due to (7.12) we get:

$$\lim_{t\to\infty}\gamma(t)\|\mathcal{U}_{R,V}(t)\|_{\mathcal{L}(L^2_x,L^2_x)}=\infty$$

and in particular due to the Banach-Steinhaus theorem the operators $\gamma(t)\mathcal{U}_{R,V}(t)$ cannot be pointwisely bounded or in an equivalent way there exists at least one $g\in L^2_x$ such that

$$\sup_{[0,\infty)}\gamma(t)\|\mathcal{U}_{R,V}(t)g\|_{L^2_x}=\infty. \tag{7.13}$$

On the other hand the function $t\to\gamma(t)\|\mathcal{U}_{R,V}(t)g\|_{L^2_x}$ is bounded on bounded sets of $[0,\infty)$ and hence (7.13) implies that

$$\limsup_{t\to\infty}\gamma(t)\|\mathcal{U}_{R,V}(t)g\|_{L^2_x}=\infty$$

which completes the proof.

Next we shall prove (7.12). Let us fix any function $f_R\in L^2_x$ such that

$$\|f_R\|_{L^2_x}\equiv 1\text{ and }\mathit{supp}\ f_R\subset\{|x|<R\}.$$

Notice that we have

$$\mathcal{U}_{R,V}(t)e^{-\mathbf{i}t\Delta_V} f_R \equiv \chi_{\{|x|<R\}} e^{\mathbf{i}t\Delta_V} e^{-\mathbf{i}t\Delta_V} f_R \equiv f_R \ \forall t \in \mathbf{R} \tag{7.14}$$

where we have used the group property of $e^{\mathbf{i}t\Delta_V}$ and the assumption done on the support of f_R.

In particular due to (1.2) and (7.14) we get:

$$\|\mathcal{U}_{R,V}(t)\|_{\mathcal{L}(L^2_x, L^2_x)} \geq \|\mathcal{U}_{R,V}(t)(e^{-\mathbf{i}t\Delta_V} f_R)\|_{L^2_x} \equiv \|f_R\|_{L^2_x} \equiv 1.$$

On the other hand (1.2) implies trivially the opposite inequality

$$\|\mathcal{U}_{R,V}(t)\|_{\mathcal{L}(L^2_x, L^2_x)} \leq 1$$

and hence (7.12) is proved.

□

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